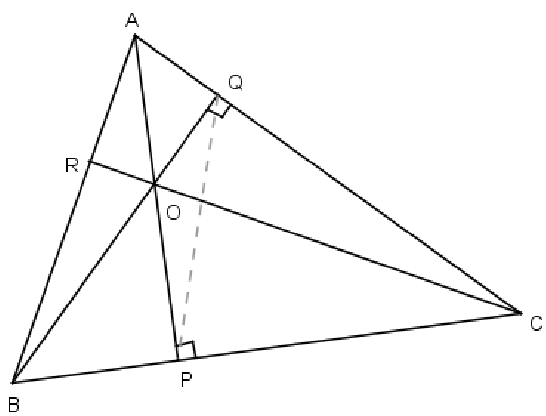
Lesson 3

In a triangle the perpendicular from a vertex to the opposite side is called an altitude.

Theorem 3.11: The altitudes of a triangle concur.

To prove this theorem we will be using some facts regarding angles in the same segment and cyclic quadrilaterals which you learned at school. Hence it is important that you review them.

For the proof let us consider the cases separately: Triangle ABC is acute, right-angled and obtuse



Let us first consider the case when ΔABC is acute

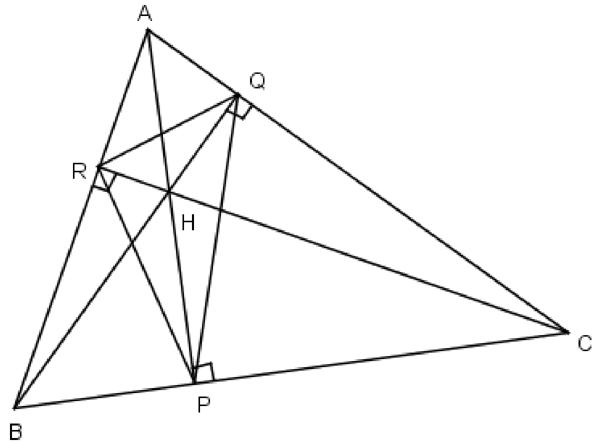
Let the feet of the altitudes from vertices A, B to their respective opposite sides be P, Q respectively. Let the altitudes AP, BQ intersect at O. Let produced CO intersect AB at R. It suffices to prove that CR is an altitude of the triangle

Proof : Join PQ. Since $\angle OPC, \angle OQC$ are right O, P, C, Q are concyclic. Since $\angle PCO (= \angle PCR)$ and $\angle PQO$ are angles in the same segment, $\angle PCO = \angle PQO$. Since $\angle BPA, \angle BQA$ are right A, B, P, Q are concyclic. Since $\angle PQB, \angle PAB$ are angles in the same segment, $\angle PCR = \angle PAR$. Hence, P, C, A, R are concyclic. Since $\angle CPA, \angle CRA$ are angles in the same segment, $\angle CPA, \angle CRA$ are right $\Rightarrow CR \perp AB$.

$Q.E.D$

The proofs for the other cases are left to the reader. The point of concurrency of the altitudes is called the Orthocenter of the triangle and is denoted usually by H.

The triangle whose vertices are the feet of the altitudes in a triangle is called the pedal triangle of the original triangle.



In the figure, ΔPQR is the pedal triangle of ΔABC.

Theorem 3.12: The altitudes of an acute angled triangle bisect the angles of the pedal triangle

Equivalently, the altitudes of a triangle are the angle bisectors of the pedal triangle. Hence, the incenter of the pedal triangle coincides with the orthocenter of the original triangle.

We will prove that AP, BQ, CR bisect $\angle RPQ, \angle PQR, \angle QRP$ respectively.

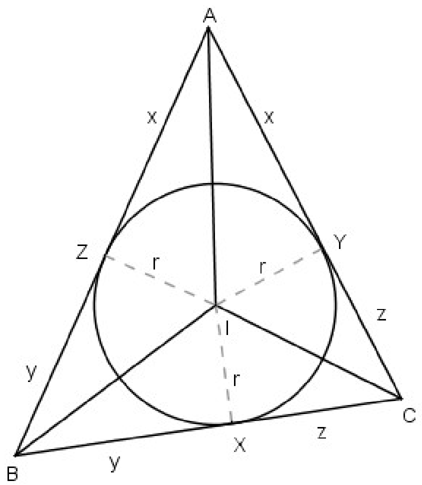
Proof : $\angle HPC = \angle HQC = 90^\circ \Rightarrow HPCQ$ is cyclic. Hence $\angle HPQ = \angle HCQ$ (Angles in the same segment). $\angle HPB = \angle HRB = 90^\circ \Rightarrow\ HRBP$ is cyclic $\Rightarrow \angle HPR = \angle HBR$ (Angles in the same segment). $\angle BQC = \angle BRC = 90^\circ \Rightarrow$ $BRQC$ is cyclic $\Rightarrow \angle HCQ = \angle HPR$ (Angles in the same segment) $\Rightarrow AP$ bisects $\angle RPQ$. Similarly, we can prove that BQ, CR bisect $\angle PQR, \angle QRP$ respectively.

$Q.E.D$

Now let us see some relationships between triangle related circles and their sides.

You already know that the internal angle bisectors of a triangle concur and that this point is equidistant from the sides. Hence there exists a circle with center at this point of concurrency and such that the sides are tangents. This circle is called the incircle of the triangle and its center, the incenter of the triangle.

Consider ΔABC . Let the lengths of the sides BC, CA, AB be a, b, c respectively (That is, denote the length of the side opposite to an angle by its corresponding lowercase letter). Let the incircle touch BC, CA, AB at X, Y, Z respectively.



Since the tangents from a point outside a circle are equal in length, $AY = AZ, BZ = BX, CX = CY$. Let $AY = AZ = x, BZ = BX = y, CX = CY = z$. We have $y + z = a, z + x = b, x + y = c \Rightarrow 2x + 2y + 2z = a + b + c$. If s is the semi perimeter of the triangle, $2x + 2y + 2z = 2s \Rightarrow x + y + z = s \Rightarrow x = s - (y + z) = s - a$. Similarly $y = s - b, z = s - c$.

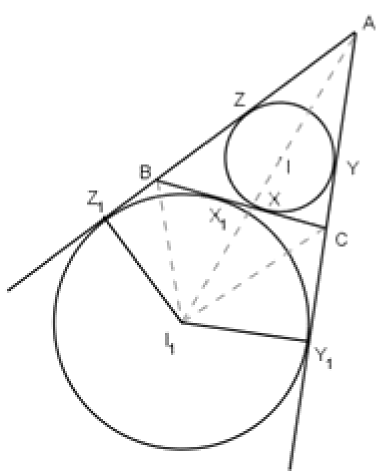
Denote the area of ΔABC by $[ABC]$. I and r are the incenter and inradius.

$$[ABC] = [IBC] + [ICA] + [IAB] = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}(a + b + c)r = sr \; (\because \; [IBC] = \frac{1}{2} \cdot BC \cdot r)$$

Theorem 3.31: The area of a triangle is equal to the product of its semi perimeter and inradius

In a triangle, we know that the external angle bisectors of two angles and the internal bisector of the other angle concur and that this point is equidistant from the sides. Hence there exists a circle with center at this point of concurrency and such that the sides (produced if necessary) are tangents. This circle is called an excircle of the triangle and its center is an excenter of the triangle. We see that there exist three such circles for a triangle.

Now consider ΔABC.



Let BC, CA, AB touch the incircle at X, Y, Z respectively. Let the excircle corresponding to vertex *A* touch BC at X_1 and produced AB, AC at Z_1, Y_1 respectively. As before, let the lengths of the sides BC, CA, AB be a, b, c respectively and the semi perimeter be s. Since the tangents from a point outside a circle are equal in length, $AZ_1 + AY_1 = AB + BZ_1 + AC + CY_1 = AB + BX_1 + CX_1 + AC = AB + BC + CA = a + b + c = 2s$. Since $AZ_1 = AY_1$, $AZ_1 = AY_1 = s$. Hence the length of the tangent from a vertex of a triangle to the opposite excircle is s.
$BZ_1 = AZ_1 - AB = s - c \Rightarrow BZ_1 = BX_1 = s - c$, $CY_1 = CX_1 = s - b$. $ZZ_1 = AZ_1 - AZ = s - (s - a) = a$.
Similarly, $YY_1 = a$. $CX = CY, CX_1 = CY_1 \Rightarrow CX + CX_1 = CY + CY_1 = YY_1 = a$. Also, $CX_1 + BX_1 = a \Rightarrow CX = BX_1$.
Similarly $BX = CX_1$. Let the excenter and its radius be I_1, r_1.
$[ABC] = [ABI_2] + [ACI_2] - [BCI_2] = \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1 = \frac{1}{2}r_1(c + b - a) = \frac{1}{2}r_1(2s - 2a) = r_1(s - a)$

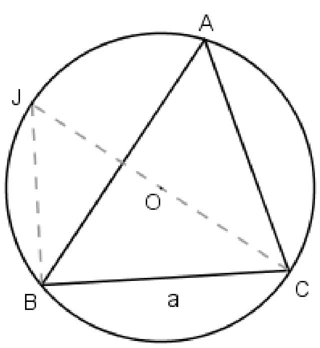
In the above line there is a typing error: It should be I_1 not I_2 .

The sine rule you learn at school is a useful result for solving many geometry problems. But, in most schools, the sine rule is not taught in its more general form.

Theorem 3.14: In ΔABC let the lengths of the sides BC, CA, AB be a, b, c respectively. Then

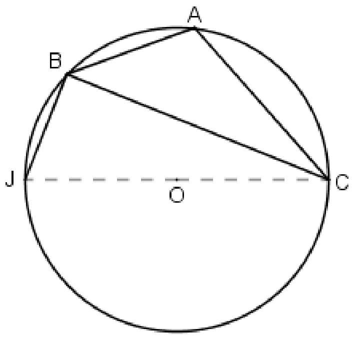
$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$ **where R is the circumradius.**

Let us first consider the case when ΔABC is acute. We know that there exists one and only one circle that passes through the vertices of ΔABC. Also, since the triangle is acute, the circumcenter lies in the interior of the triangle. Denote the circumcenter by O.



Proof: Draw the diameter CJ and join BJ .Then $\angle CBJ$ is right $\Rightarrow \Delta CBJ$ is a right triangle $\Rightarrow sinJ = \frac{BC}{CJ} = \frac{a}{2R}$. Since $\angle BJC, \angle BAC$ are angles in the same segment, $\angle BJC = \angle BAC \Rightarrow sinA = \frac{a}{2R}$. Similarly, we can show that $sinB = \frac{b}{2R}$, $sinC = \frac{c}{2R}$.

Now consider the case : ΔABC is obtuse. In this case the circumcenter lies outside the triangle.
Draw the diameter CJ and join BJ .



Since $\angle CBJ$ is right, ΔCBJ is a right triangle $\Rightarrow sinJ = \frac{BC}{CJ} = \frac{a}{2R}$. $ABJC$ is cyclic $\Rightarrow \angle A + \angle J = 180^\circ \Rightarrow \angle J = 180^\circ - \angle A \Rightarrow sinJ = \sin(180^\circ - A) = sinA \Rightarrow sinA = \frac{a}{2R}$. Similarly, we can prove that $sinB = \frac{b}{2R}, sinC = \frac{c}{2R}$.

The case ΔABC is right triangle can be easily proved.

$Q.E.D$

Problems

1. Let P be the center of the square erected externally on the hypotenuse AC of right triangle ABC . Prove that BP bisects $\angle ABC$

2. Prove that the orthocenter of the medial triangle coincides with the circumcenter of the original triangle

3. Q is a point located in the interior of parallelogram ABCD such that $\angle QBC = \angle QDC = 90^\circ$. Prove that $BD \perp AQ$

4. In triangle ABC, E is the midpoint of BC and F is a point on AC such that AC=3FC. Find the ratio of the areas of ΔFEC and quadrilateral $ABEF$

5. O is a point in the interior of square ABCD such that $\angle ODC = \angle OCD = 15^\circ$. Prove that triangle OBA is equilateral.

Send us your solutions of some or all of the problems to the below address before the 7th of October 2009