IMC Selection Test - 2019

May 10, 2019 - 9:00am to 11:30am

Part A : Short Answer Questions

Write only the final answer in the answer sheet provided. Each question is worth 5 points. No partial credit will be awarded.

1. Find all triples of real numbers (x, y, z) such that

$$x\sqrt{y-1} + y\sqrt{z-1} + z\sqrt{x-1} \geq \frac{xy + yz + zx}{2}$$

2. Let ABC be an isosceles triangle with $AB = AC$, $BC < AC$ and $\angle A = 20^0$. Point D is chosen on the side AC such that $AD = BC$. Determine the value of $\angle BDC$ in degrees.

3. Let A be a non-empty subset of $\mathbb{R}$ such that the following are true:

   (a) A has at most 5 elements.

   (b) If $x \in A$ then $\dfrac{1}{x} \in A$ and $1 - x \in A$.

   Find all such subsets.

4. In a 100×100 square grid each 1×1 square is to be painted either black or red such that in any 4×4 grid exactly two squares are black. Surath has already painted the first two squares in the first row black. In how many different ways can Surath finish painting the array?

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x - f(y)) = 1 - x - y$ for every $x, y \in \mathbb{R}$.

6. Let $ABCDE$ be a regular pentagon such that the star region $ACEBDA$ (i.e pentagram $ABCDE$) has area 1. Let the point of intersection of AC and BE be P, and let the point of intersection of BD and CE be Q. Determine the area of the quadrilateral $APQD$.

7. For positive integers n and d let $r(n, d)$ denote the smallest positive remainder when n is divided by d. Let $S(n) = \sum_{d=1}^{n} r(n, d)$. Determine the value of $S(2020) - S(2019)$.

8. Let p, q be distinct primes and $A = \{p^m q^n : m, n \in \{0, 1, 2, 3, ..., 2019\}\}$. Determine the least number k such that every subset of A consisting of k elements contains a pair of elements x, y where x is a divisor of y.

Section B: Essay Type

Full written solutions are required. Use the booklets provided to write the answers. Use a separate booklet for each question. Each question is worth 20 points.

1. There are 2019 stones in a bag. Upul and Madara take turns in removing stones from the bag in such a way that in each turn the number of stones removed is at least one and at most half of the number of stones in the bag. For example the number of stones that can be removed from the bag in the first turn is between 1 and 1009, including 1 and 1009. The player who leaves just one stone in the bag looses the game. Madara takes the first turn. Which player has a winning strategy? Justify your answer.

2. In the triangle ABC, B_1 and C_1 are points on the sides AC and AB, respectively. Lines BB_1 and CC_1 intersect at point D. Prove that a circle can be inscribed inside quadrilateral AB_1DC_1 if and only if the incircles of the triangles ABD and ACD are tangent to each other.

3. Is it possible to find 16 three-digit numbers greater that 99 such that both the statements (a) and (b) given below are true?

   (a) The total number of distinct digits used in the 16 numbers is three.

   (b) No two of them have the same remainder when divided by 16.